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Perturbed Coulomb potentials in the Klein–Gordon equation via the shifted-l expansion technique

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Abstract. A shifted-l expansion technique is introduced to calculate the energy eigenvalues for the Klein–Gordon (KG) equation with Lorentz vector and/or Lorentz scalar potentials. Although it applies to any spherically symmetric potential, those that include Coulomb-like terms are only considered. Exact eigenvalues for a Lorentz vector or a Lorentz scalar, and an equally mixed Lorentz vector and Lorentz scalar coulombic potentials are reproduced. Highly accurate and rapidly converging ground-state energies for Lorentz vector Coulomb with a Lorentz vector or a Lorentz scalar linear potential, $V(r) = -A_1/r + kr$, and $V(r) = -A_1/r$ and $S(r) = kr$, respectively, are obtained. Moreover, a simple straightforward closed-form solution for a KG-particle in coulombic Lorentz vector and Lorentz scalar potentials is presented.

1. Introduction

The Klein–Gordon (KG) and the Dirac equations with Lorentz scalar (added to the mass term) and/or Lorentz vector (coupled as the 0-component of the four-vector potential) potentials are of interest in many branches of physics. For example, Lorentz scalar or equally mixed Lorentz scalar and Lorentz vector potentials have considerable interest in quark–antiquark mass spectroscopy [1–5]. Lorentz vector potentials have great utility in atomic, nuclear and plasma physics [6, 7]. Therefore many attempts have been made to develop approximation techniques to treat relativistic particles in the KG and Dirac equations [1–7].

Very recently we introduced a shifted-l expansion technique (SLET) to solve the Schrödinger [8], and Dirac equations for some model potentials [9]. SLET is a reformation to the existing shifted-N expansion technique (SLNT) [1, 10–12] and references therein. SLET simply consists of using $1/\bar{l}$ as an expansion parameter where $\bar{l} = l - \beta$, β is a suitable shift, l is the angular momentum quantum number for spherically symmetric potentials, and $l = |m|$ for cylindrically symmetric potentials, where m is the magnetic quantum number. As such, one does not need to construct the N-dimensional form, required to perform SLNT, of the wave equation of interest. With SLET we simply expand through the quantum number in the centrifugal term of that equation. Unlike other perturbation methods [13–16], SLET puts no constraints on the coupling constants of the potential or on the quantum numbers involved. Above all, it yields very accurate and rapidly converging eigenvalues without the need of wavefunctions or matrix elements.

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In this paper we shall be concerned with the shifted-l expansion for the KG equation with radially symmetric Lorentz scalar, $S(r)$, and/or Lorentz vector, $V(r)$, potentials that include Coulomb-like terms. We shall examine SLET and calculate the energy eigenvalues for the KG equation with the following potential mixtures. (i) $V(r) = -A_1/r$ and $S(r) = 0$, which represents a π^- meson in a Coulomb potential. (ii) $V(r) = 0$ and $S(r) = -A_2/r$, which has no experimental evidence, to the best of our knowledge, thus our calculations are only of academic interest. (iii) $V(r) = S(r) = -A/r$ which represents not only a KG-particle in an equally mixed Lorentz scalar and Lorentz vector potential but also a Dirac particle in the same potential mixture, where $l = j + \frac{1}{2}$ and the radial KG wavefunction represents the radial large component of the Dirac spinor [4, 9]. (iv) $V(r) = -A_1/r + kr$ and $S(r) = 0$ representing a π^- meson in a Coulomb potential perturbed by a linear Lorentz vector interaction kr. (v) $V(r) = -A_1/r$ and $S(r) = kr$ describing a π^- meson in a Coulomb potential perturbed by a linear Lorentz scalar potential kr.

In section 2 we shall introduce SLET for the KG equation with any spherically symmetric Lorentz scalar and/or Lorentz vector potentials that include Coulomb-like interactions. We shall cast SLET's analytical expressions in such a way that allows the reader to use them without proceeding into their derivations. In section 3 we shall show that these expressions yield closed-form solutions to the KG equation for mixtures (i)–(iii). Ground-state energies for mixtures (iv) and (v) will be calculated and compared with those of McQuarrie and Vrscay [13] in the same section. We conclude and make remarks in section 4.

In appendix A we present a simple straightforward closed-form solution for the KG equation with Coulomb-like Lorentz scalar and Lorentz vector potentials. It could be interesting to mention that a similar solution was found by McQuarrie and Vrscay [13], who used a confluent hypergeometric function in their calculation. They have misprinted it though (see the appendix of [13]). To the best of our knowledge, such explicit solution has not been reported elsewhere.

2. SLET for the KG equation with potentials including coulombic terms

In this section we shall consider the three-dimensional KG equation with radially symmetric Lorentz vector and Lorentz scalar potentials, $V(r)$ and $S(r)$, respectively. If $\Psi(\boldsymbol{r})$ denotes the wavefunction of the KG particle, a separation of variables $\Psi(r) = r^{-1}R(r)Y(\theta, \phi)$ yields the following radial equation (in units $\hbar = c = 1$) [1]:

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + [S(r) + m]^2 - [E - V(r)]^2\right] R(r) = 0 \tag{1}$$

where E is the energy and l is the angular quantum number. For Coulomb-like potentials one may use the substitutions:

$$V_r(r) = V(r)^2 - A_1^2/r^2 \tag{2}$$

and

$$S_r(r) = S(r)^2 - A_2^2/r^2 \tag{3}$$

so that equation (1) becomes

$$\left[-\frac{d^2}{dr^2} + \frac{l'(l'+1)}{r^2} + \gamma(r) + 2EV(r)\right] R(r) = E^2 R(r) \tag{4}$$

where

$$\gamma(r) = -V_r(r) + S_r(r) + 2mS(r) + m^2 \tag{5}$$

$$l'(l'+1) = l(l+1) - A_c \qquad l' = -\tfrac{1}{2} + \sqrt{(l+1/2)^2 - A_c} \tag{6}$$

$$A_c = A_1^2 - A_2^2. \tag{7}$$

Hereby, it should be noted that for the case of $V(r) = -A_1/r$ and $S(r) = -A_2/r$, equation (4) reduces to a form nearly identical to the Schrödinger equation for a Coulomb field. Its solution can thus be inferred from the known solution of the Schrödinger–Coulomb problem. We carried this out in appendix A.

If we shift l' through the relation $l' = \bar{l} + \beta$, equation (4) reads

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{[\bar{l}^2 + \bar{l}(2\beta+1) + \beta(\beta+1)]}{r^2} + \gamma(r) + 2EV(r)\right] R(r) = E^2 R(r) \tag{8}$$

where β is a suitable shift to be determined and is mainly introduced to avoid the trivial case when $l' = 0$.

To start the systematic $1/\bar{l}$ expansion [8, 9] we define

$$\gamma(r) = \frac{\bar{l}^2}{Q}[\gamma(r_0) + \gamma'(r_0)r_0 x/\bar{l}^{1/2} + \gamma''(r_0)r_0^2 x^2/2\bar{l} + \cdots] \tag{9}$$

$$V(r) = \frac{\bar{l}^2}{Q}[V(r_0) + V'(r_0)r_0 x/\bar{l}^{1/2} + V''(r_0)r_0^2 x^2/2\bar{l} + \cdots] \tag{10}$$

$$E = \frac{\bar{l}^2}{Q}[E_0 + E_1/\bar{l} + E_2/\bar{l}^2 + E_3/\bar{l}^3 + \cdots] \tag{11}$$

where $x = \bar{l}^{1/2}(r - r_0)/r_0$, r_0 is currently an arbitrary point to perform Taylor expansions about, with its particular value to be determined below, and Q is to be set equal to $\bar{l}^2$ at the end of the calculations. Substituting equations (9)–(11) into equation (8) implies

$$\begin{aligned}
&\left[\frac{-\mathrm{d}^2}{\mathrm{d}x^2} + \left(\bar{l} + (2\beta+1) + \frac{\beta(\beta+1)}{\bar{l}}\right)\left(1 - \frac{2x}{\bar{l}^{1/2}} + \frac{3x^2}{\bar{l}} - \cdots\right)\right.\\
&\qquad + \frac{r_0^2\bar{l}}{Q}\left(\gamma(r_0) + \frac{\gamma'(r_0)r_0 x}{\bar{l}^{1/2}} + \frac{\gamma''(r_0)r_0^2x^2}{2\bar{l}} + \frac{\gamma'''(r_0)r_0^3x^3}{6\bar{l}^{3/2}} + \cdots\right)\\
&\qquad \left. + \frac{2r_0^2\bar{l}}{Q}\left(V(r_0) + \frac{V'(r_0)r_0 x}{\bar{l}^{1/2}} + \cdots\right)\left(E_0 + \frac{E_1}{\bar{l}} + \frac{E_2}{\bar{l}^2} + \cdots\right)\right]\Phi_{n_r}(x)\\
&\quad = \mu_{n_r}\Phi_{n_r}(x)
\end{aligned} \tag{12}$$

where

$$\mu_{n_r} = \frac{r_0^2\bar{l}}{Q}\left[E_0^2 + \frac{2E_0E_1}{\bar{l}} + \frac{(E_1^2 + 2E_0E_2)}{\bar{l}^2} + \frac{2(E_0E_3 + E_1E_2)}{\bar{l}^3} + \cdots\right] \tag{13}$$

and n_r is the radial quantum number. Equation (12) is a Schrödinger-like equation for the one-dimensional anharmonic oscillator and has been discussed in detail by Imbo *et al* [11]. We therefore quote only the resulting eigenvalue of [11] and write

$$\begin{aligned}
\mu_{n_r} = \bar{l}&\left[1 + \frac{2r_0^2V(r_0)E_0}{Q} + \frac{r_0^2\gamma(r_0)}{Q}\right] + \left[(2\beta+1) + \frac{2r_0^2V(r_0)E_1}{Q} + \left(n_r + \frac{1}{2}\right)w\right]\\
&+ \frac{1}{\bar{l}}\left[\beta(\beta+1) + \frac{2r_0^2V(r_0)E_2}{Q} + \alpha_1\right] + \frac{1}{\bar{l}^2}\left[\frac{2r_0^2V(r_0)E_3}{Q} + \alpha_2\right]
\end{aligned} \tag{14}$$

where α_1 and α_2 are given in appendix B of this paper. If we compare equation (14) with (13), we obtain

$$E_0 = V(r_0) \pm \sqrt{V(r_0)^2 + Q/r_0^2 + \gamma(r_0)} \tag{15}$$

$$E_1 = \frac{Q}{2r_0^2(E_0 - V(r_0))}\left[2\beta + 1 + \left(n_r + \frac{1}{2}\right)w\right] \tag{16}$$

$$E_2 = \frac{Q}{2r_0^2(E_0 - V(r_0))}[\beta(\beta+1) + \alpha_1] \tag{17}$$

$$E_3 = \frac{Q}{2r_0^2(E_0 - V(r_0))}\alpha_2 \tag{18}$$

and

$$E_{n_r} = E_0 + \frac{1}{2r_0^2(E_0 - V(r_0))}\left[\beta(\beta+1) + \alpha_1 + \frac{\alpha_2}{\bar{l}}\right]. \tag{19}$$

Here r_0 is chosen to be the minimum of E_0, i.e.

$$\frac{\mathrm{d}E_0}{\mathrm{d}r_0} = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2E_0}{\mathrm{d}r_0^2} > 0. \tag{20}$$

Hence, r_0 is obtained through the relation

$$2Q = 2(l' - \beta)^2 = b(r_0) + \sqrt{b(r_0)^2 - 4c(r_0)} \tag{21}$$

where

$$b(r_0) = r_0^3[2V(r_0)V'(r_0) + \gamma'(r_0) + r_0V'(r_0)^2] \tag{22}$$

$$c(r_0) = \frac{r_0^6}{4}[\gamma'(r_0)^2 + 4V(r_0)V'(r_0)\gamma'(r_0) - 4\gamma(r_0)V'(r_0)^2]. \tag{23}$$

The shifting parameter β is determined by requiring $E_1 = 0$ [1, 8–12] to obtain

$$\beta = -[1 + (n_r + \tfrac{1}{2})w]/2 \tag{24}$$

where

$$w = \left[12 + \frac{2r_0^4\gamma''(r_0)}{Q} + \frac{4r_0^4V''(r_0)E_0}{Q}\right]^{1/2}. \tag{25}$$

It is convenient to summarize the above procedure in the following steps. (a) Calculate Q from equation (21) and substitute it into equation (15) to find E_0 in terms of r_0. (b) Substitute E_0 and Q into equation (25) to obtain w. (c) Find β from equation (24) to calculate r_0 from equation (21). (e) Finally, one can obtain E_0 and calculate E_{n_r} from equation (19). However, one is not always able to calculate r_0 in terms of the potential coupling constants since the analytical expressions become algebraically complicated, although straightforward. Therefore, one has to appeal to numerical computations to find r_0 and hence E_0.

3. Applications, results and discussion

To show the performance of the analytical expressions of SLET it is best to consider some special cases.

3.1. $V(r) = -A_1/r$ and $S(r) = 0$

A pionic atom in a Coulomb potential obeys the KG equation with $V(r) = -A_1/r$ and $S(r) = 0$. To calculate its bound-state energies, which are simply the bound-state energies of a π^- meson in a Coulomb potential, we follow the SLET procedure and find

$$E_0 = \frac{-A_1^2 \pm (A_1^2 + Q)}{A_1 r_0}. \tag{26}$$

Here we have to choose the positive sign since states with negative energies correspond to antiparticles. Furthermore, the negative sign yields a contradiction to equation (21). Hence $w = 2$,

$$Q = (l' - \beta)^2 = \left[n_r + \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 - A_1^2}\right]^2 \tag{27}$$

$$r_0 = \sqrt{\frac{Q^2 + QA_1^2}{m^2 A_1^2}} \tag{28}$$

and

$$E_0 = m\left[1 + \frac{A_1^2}{\tilde{n}^2}\right]^{-1/2} \tag{29}$$

where $\tilde{n} = \sqrt{Q}$. Equation (29) represents the well known closed-form solution of the KG equation for a π^- meson in a Coulomb potential [17]. It should be noted that higher-order terms of the energy eigenvalues vanish identically, i.e. $E_2 = 0$ and $E_3 = 0$. Hence $E_{n_r} = E_0$.

3.2. $V(r) = 0$ and $S(r) = -A_2/r$

Since there is no experimental evidence, to the best of our knowledge, for such a long-range interaction, our calculations are only of academic interest. Following the above procedure we find

$$r_0 = \frac{(l' - \beta)^2}{mA_2} \tag{30}$$

and

$$E_0 = \pm m\left[1 - \frac{A_2^2}{\tilde{n}^2}\right]^{1/2} \tag{31}$$

where $\tilde{n} = n_r + \frac{1}{2} + \sqrt{(l + \frac{1}{2})^2 + A_2^2}$. Again the higher-order terms of the energy eigenvalues vanish identically. Thus $E_{n_r} = E_0$.

Obviously there exist two branches of solutions in the bound region and they exhibit identical behaviour, which reflects the fact that the Lorentz scalar interaction does not distinguish between particles and antiparticles. The particle and antiparticle states, positive and negative energies, respectively, approach each other with increasing coupling constant, without touching. Therefore, spontaneous pair creation never occurs, no matter how strong the potential chosen.

3.3. $V(r) = S(r) = -A/r$

This type of potential mixture, $V(r) = S(r)$, has considerable interest in quarkonium spectroscopy [4, 9, 18]. For the particular case $V(r) = S(r) = -A/r$, SLET yields

$$E_0 = -A/r_0 \pm m \tag{32}$$

and

$$(E_0^2 - m^2)r_0^2 = A^2 \mp (Q + A^2). \tag{33}$$

Equation (33) can be satisfied if and only if the negative sign is chosen, otherwise it contradicts equation (21). The only valid sign in equation (32) is thus the positive one, and hence

$$E_0 = -A/r_0 + m \tag{34}$$

which in turn implies that

$$r_0 = \frac{n^2 + A^2}{2mA} \qquad n = n_r + l + 1 \tag{35}$$

and

$$E_{n_r} = E_0 = m\left[1 - \frac{2A^2}{n^2 + A^2}\right] \tag{36}$$

where higher-order terms of the energy eigenvalues vanish identically, and n is the principle quantum number. For $A \to \infty$, E_{n_r} approaches the value $-m$ asymptotically, but the state never goes into the negative continuum.

To show that equation (36) yields the energy eigenvalues for Dirac particle in the same potential mixture, we replace l by $j + \frac{1}{2}$ to obtain

$$E_{n_r} = m\left[1 - \frac{2A^2}{(n_r + |\kappa| + 1)^2 + A^2}\right] \tag{37}$$

where $|\kappa| = j + \frac{1}{2}$ [19].

3.4. $V(r) = -A_1/r + kr$ *and* $S(r) = 0$

This potential represents a π^- meson in a Coulomb potential perturbed by a linear Lorentz vector potential kr. In this case

$$\gamma(r) = -k^2r^2 + 2A_1k + m^2. \tag{38}$$

Equation (38), when substituted in (21), (15), (25), (24), and again in (21), respectively, yields a very involved algebraic equation for r_0. We solve this equation numerically with a maximum error of order $\sim 10^{-15}$ to calculate for r_0. Once r_0 is calculated, Q, E_0, w, β, and hence E_{n_r} can be obtained.

In tables 1 and 2 we list our results for the ground-state energies along with those of McQuarrie and Vrscay [13], who used hypervirial and Hellmann–Feynman theorems to construct Rayleigh–Schrödinger (RS) perturbation expressions to an arbitrary order. Our results are given in such a way that the contributions of the second- and third-order corrections, $E_2/\bar{l}^2$ and $E_3/\bar{l}^3$, respectively, to the energy eigenvalues are made clear. The results are in excellent agreement with those of [13].

Table 1. Ground-state energies of a π^- meson in $V(r) = -A_1/r + kr$ and $S(r) = 0$ (in $\hbar = c = m = 1$ units). The lower bounds to the energies E of [13] are obtained by replacing the last j digits of the upper bounds with the j digits in parentheses.

A_1	k	[13]	E_0	$E_0 + E_2/\bar{l}^2$	Equation (11)
0.2	0.0	0.978 906 312 93	0.978 906 312 93	0.978 906 312 93	0.978 906 312 93
	0.01	1.027 622(19)	1.029 590	1.027 995	1.027 641
	0.05	1.152(48)	1.154 1	1.151 4	1.150 4
	0.1	1.277(48)	1.268 1	1.264 8	1.263 4
	0.2	1.50(37)	1.447 8	1.443 6	1.441 6
	0.3	1.73(45)	1.595 9	1.590 7	1.588 2
0.3	0.0	0.948 683 298 1	0.948 683 298 1	0.948 683 298 1	0.948 683 298 1
	0.01	0.984 379 538 0(78)	0.986 139 171 3	0.984 457 889 0	0.984 383 623 7
	0.05	1.086 12(08)	1.091 60	1.087 10	1.086 11
	0.1	1.183 98(08)	1.191 11	1.185 17	1.183 56
	0.2	1.345(34)	1.350 0	1.342 1	1.339 7
	0.3	1.487(52)	1.482 0	1.472 3	1.469 3

Table 2. Ground-state energies of a π^- meson in $V(r) = -A_1/r + kr$ and $S(r) = 0$ (in $\hbar = c = m = 1$ units). The lower bounds to the energies E of [13] are obtained by replacing the last j digits of the upper bounds with the j digits in parentheses.

A_1	k	[13]	E_0	$E_0 + E_2/\bar{l}^2$	Equation (11)
0.4	0.0	0.894 427 191	0.894 427 191	0.894 427 191	0.894 427 191
	0.01	0.919 049 561 9	0.919 938 910 5	0.919 018 589 9	0.919 059 255 7
	0.05	0.997 350 23(19)	1.002 924 92	0.997 639 76	0.997 325 40
	0.1	1.075 966 6(05)	1.085 037 30	1.076 893 28	1.075 856 18
	0.2	1.204 88(59)	1.218 25	1.206 62	1.204 53
	0.3	1.313 8(20)	1.329 87	1.315 51	1.312 61
0.5	0.0	0.707 106 781 19	0.707 106 781 19	0.707 106 781 19	0.707 106 781 19
	0.01	0.717 444 184 5	0.717 581 677 8	0.717 439 534 4	0.717 444 606 7
	0.05	0.754 810 427 9	0.756 979 656 2	0.754 611 694 0	0.754 864 528 5
	0.1	0.795 714 744 277(07)	0.801 365 443 6	0.795 133 211 2	0.795 908 073 8
	0.2	0.866 135 31(67)	0.878 413 86	0.865 081 53	0.866 484 55
	0.3	0.926 9(68)	0.945 0	0.925 86	0.927 34

3.5. $V(r) = -A_1/r$ *and* $S(r) = kr$

A π^- meson in a Coulomb potential perturbed by a linear scalar interaction is described by $V(r) = -A_1/r$ and $S(r) = kr$ potential mixture in the KG equation. In this case

$$\gamma(r) = k^2r^2 + 2mkr + m^2. \tag{39}$$

Following the same steps as (iv), we numerically solve for r_0, to a maximum error of order $\sim 10^{-15}$, Q, E_0, w, β, and E_{n_r}. Our results for the ground-state energies are presented in tables 3 and 4 in such a way that the convergence of SLET is made clear. We compare them with those of [13]; they are in excellent agreement.

In view of the above results, the following observations deserve to be recorded.

The closed-form solutions, equations (29), (31) and (36), being obtained by the leading term of the energy series, equation (11), where higher-order terms vanished identically, reveals how rapidly converging are the results of SLET.

Table 3. Ground-state energies of a π^- meson in $V(r) = -A_1/r$ and $S(r) = kr$ (in $\hbar = c = m = 1$ units). The lower bounds to the energies E of [13] are obtained by replacing the last j digits of the upper bounds with the j digits in parentheses.

A_1	k	[13]	E_0	$E_0 + E_2/\bar{l}^2$	Equation (11)
0.2	0.0	0.978 906 312 9	0.978 906 312 9	0.978 906 312 9	0.978 906 312 9
	0.01	1.026 683 9(09)	1.028 715 02	1.027 085 79	1.026 712 48
	0.05	1.145 795(48)	1.147 917 0	1.144 985 3	1.143 939 3
	0.1	1.263(34)	1.254 3	1.250 6	1.249 1
	0.2	1.47(34)	1.418	1.413	1.411
	0.3	1.68(41)	1.551	1.545	1.542
0.3	0.0	0.948 683 298 05	0.948 683 298 05	0.948 683 298 05	0.948 683 298
	0.01	0.983 411 945 0(49)	0.985 263 214 6	0.983 516 474 8	0.983 411 820 1
	0.05	1.079 797(62)	1.085 772	1.081 002	1.079 875
	0.1	1.170(69)	1.178 26	1.171 82	1.169 99
	0.2	1.316(05)	1.322 7	1.314 03	1.311 33
	0.3	1.443(09)	1.440 4	1.429 98	1.426 66

Table 4. Ground-state energies of a π^- meson in $V(r) = -A_1/r$ and $S(r) = kr$ (in $\hbar = c = m = 1$ units). The lower bounds to the energies E of [13] are obtained by replacing the last j digits of the upper bounds with the j digits in parentheses.

A_1	k	[13]	E_0	$E_0 + E_2/\bar{l}^2$	Equation (11)
0.4	0.0	0.894 427 191	0.894 427 191	0.894 427 191	0.894 427 191
	0.01	0.918 077 922 7	0.919 060 388 2	0.918 053 799 8	0.918 087 882 4
	0.05	0.991 505 0(49)	0.997 801 768	0.992 043 95	0.991 507 96
	0.1	1.063 490(84)	1.073 956 16	1.065 030 89	1.063 524 24
	0.2	1.179 1(88)	1.194 92	1.182 12	1.179 26
	0.3	1.275 5(37)	1.294 49	1.278 83	1.275 03
0.5	0.0	0.707 106 781 2	0.707 106 781 2	0.707 106 781 2	0.707 106 781 2
	0.01	0.716 815 172 3	0.716 999 889 9	0.716 809 667 5	0.716 815 725 2
	0.05	0.751 435 153 8	0.754 423 303 6	0.751 234 278 4	0.751 497 637 7
	0.1	0.788 775 203 61(59)	0.796 708 110 1	0.788 336 499 1	0.788 958 858 8
	0.2	0.852 303 690(62)	0.869 749 249	0.852 125 792	0.852 493 296
	0.3	0.906 822 3(16)	0.932 267 3	0.907 442 1	0.906 889

The numerical results of SLET, in tables 1–4, imply that the contributions of the second- and third-order corrections to the energy eigenvalues are almost negligible. The convergence of SLET is thus out of the question. However, the RS coefficients E^p for the eigenvalue

$$E = \sum_{p=0}^{\infty} E^{(p)} k^p \tag{40}$$

used in [13], as well as their continued-fraction (CF) counterparts c_p were computed numerically to large order, $p \sim 100$ and $p \sim 50$ for the Lorentz vector linear and the Lorentz scalar linear perturbations, respectively. Numerical ratio tests showed that the perturbation series are divergent [13]. Also, the c_p had suffered from occasional eruptions reversing the roles of the upper and lower bounds of the energy eigenvalues. Moreover, the gap between the two bounds increases with increasing coupling constant k, as does the uncertainty of the energy eigenvalues.

4. Conclusions and remarks

In this paper we have introduced SLET to solve the eigenvalues of KG equation with Lorentz vector and Lorentz scalar potentials including coulombic terms. Although it applies to any spherically symmetric potential, those that include Coulomb-like terms were only considered. We have reproduced closed-form solutions for a Lorentz vector or a Lorentz scalar, and for an equally mixed Lorentz vector and Lorentz scalar coulombic potentials [20]. Compared with those of [13] our results are highly accurate and rapidly convergent.

The conceptual soundness of our SLET is obvious. It is highly accurate and efficient with respect to computer time. It does not need the wavefunctions or matrix elements, but when necessary wavefunctions can be calculated. It puts no constraints on the coupling constants of the potential or on the quantum numbers. It simply consists of using $1/\bar{l}$ as an expansion parameter rather than the coupling constant of the potential. It is to be understood as being an expansion through not only the angular momentum quantum number but also through any existing quantum number in the centrifugal-like term of any Schrödinger-like equation, equation (4).

A general observation concerning the method used by McQuarrie and Vrscay [13] is in order. Unlike our approach, their method involves expansions through the coupling constant k, equation (40). Thus, whereas their computations for the ground-state energies are beyond doubt, the same need not be true for the case of strong coupling constant $k > 1$, in equation (40), for example.

Finally, we would like to remark that SLET is also applicable to more complicated potentials. For example, the screened Coulomb potentials which have great utility in atomic, nuclear and plasma physics. The equally mixed Lorentz scalar and Lorentz vector logarithmic potential which has significant interest in quarkonium spectroscopy [4].

Appendix A. The KG equation with Coulomb-like Lorentz scalar and Lorentz vector potentials

In this section we present a simple solution for a KG particle in Coulomb-like Lorentz scalar and Lorentz vector potentials, i.e. $V(r) = -A_1/r$ and $S(r) = -A_2/r$. For this particular problem the KG equation reduces to

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l'(l'+1)}{r^2} - \frac{2(mA_2 + EA_1)}{r}\right] R(r) = [E^2 - m^2] R(r). \tag{41}$$

It is obvious that this equation is in a form nearly identical to the Schrödinger equation for a Coulomb potential. Its solution can thus be inferred from the known Schrödinger–Coulomb solution. Therefore, one may obtain its solution through the relation

$$E^2 - m^2 = \frac{-(2mA_2 + 2EA_1)^2}{(2\tilde{n})^2}. \tag{42}$$

This equation is quadratic in E and thus admits a solution of the form

$$E_{n_r} = m\left[\frac{-A_1A_2 \pm \sqrt{A_1^2A_2^2 + (\tilde{n}^2 + A_1^2)(\tilde{n}^2 - A_2^2)}}{\tilde{n}^2 + A_1^2}\right] \tag{43}$$

where $\tilde{n} = n_r + l' + 1$. Hereby, it should be pointed out that this result reduces to those obtained in sections 3.1–3.3. Although McQuarrie and Vrscay [13] used a confluent hypergeometric function to obtain this result, they have misprinted it (see the appendix of [13]).

Appendix B. α_1 and α_2 in equation (14)

The definitions of α_1 and α_2 which appeared in equation (14) are:

$$\alpha_1 = [(1+2n_r)e_2 + 3(1+2n_r+2n_r^2)e_4] - w^{-1}[e_1^2 + 6(1+2n_r)e_1e_3 + (11+30n_r+30n_r^2)e_3^2] \tag{44}$$

$$\begin{aligned}\alpha_2 = {} & (1+2n_r)d_2 + 3(1+2n_r+2n_r^2)d_4 + 5(3+8n_r+6n_r^2+4n_r^3)d_6 - w^{-1}[(1+2n_r)e_2^2 \\ & +12(1+2n_r+2n_r^2)e_2e_4 + 2e_1d_1 + 2(21+59n_r+51n_r^2+34n_r^3)e_4^2 \\ & +6(1+2n_r)e_1d_3 + 30(1+2n_r+2n_r^2)e_1d_5 + 6(1+2n_r)e_3d_1 \\ & +2(11+30n_r+30n_r^2)e_3d_3 + 10(13+40n_r+42n_r^2+28n_r^3)e_3d_5] \\ & +w^{-2}[4e_1^2e_2 + 36(1+2n_r)e_1e_2e_3 + 8(11+30n_r+30n_r^2)e_2e_3^2 \\ & +24(1+n_r)e_1^2e_4 + 8(31+78n_r+78n_r^2)e_1e_3e_4 + 12(57+189n_r+225n_r^2 \\ & +150n_r^3)e_3^2e_4] - w^{-3}[8e_1^3e_3 + 108(1+2n_r)e_1^2e_3^2 \\ & +48(11+30n_r+30n_r^2)e_1e_3^3 + 30(31+109n_r+141n_r^2+94n_r^3)e_3^4]\end{aligned} \tag{45}$$

where

$$e_j = \frac{\varepsilon_j}{w^{j/2}} \qquad \text{and} \qquad d_i = \frac{\delta_i}{w^{i/2}} \tag{46}$$

with $j = 1, 2, 3, 4$, $i = 1, 2, 3, 4, 5, 6$, and

$$\varepsilon_1 = -2(2\beta+1) \qquad \varepsilon_2 = 3(2\beta+1) \tag{47}$$

$$\varepsilon_3 = -4 + \frac{r_0^5}{6Q}[\gamma'''(r_0) + 2V'''(r_0)E_0] \tag{48}$$

$$\varepsilon_4 = 5 + \frac{r_0^6}{24Q}[\gamma''''(r_0) + 2V''''(r_0)E_0] \tag{49}$$

$$\delta_1 = -2\beta(\beta+1) + \frac{2r_0^3V'(r_0)E_2}{Q} \tag{50}$$

$$\delta_2 = 3\beta(\beta+1) + \frac{r_0^4V''(r_0)E_2}{Q} \tag{51}$$

$$\delta_3 = -4(2\beta+1) \qquad \delta_4 = 5(2\beta+1) \tag{52}$$

$$\delta_5 = -6 + \frac{r_0^7}{120Q}[\gamma'''''(r_0) + 2V'''''(r_0)E_0] \tag{53}$$

$$\delta_6 = 7 + \frac{r_0^8}{720Q}[\gamma''''''(r_0) + 2V''''''(r_0)E_0]. \tag{54}$$

The terms including E_1 have been dropped from the expressions above since $E_1 = 0$.

References

[1] Mustafa O and Sever R 1991 *Phys. Rev.* A **43** 5787
Mustafa O and Sever R 1991 *Phys. Rev.* A **44** 4142
[2] Papp E 1991 *Ann. Phys., Lpz.* **48** 319
Papp E 1991 *Phys. Lett.* B **259** 19
[3] Stepanov S and Tutik R 1992 *Phys. Lett.* A **163** 26
Roychoudhury R and Varshni Y 1987 *J. Phys. A: Math. Gen.* **20** L1083
[4] Critchfield C 1976 *J. Math. Phys.* **17** 261
Gunion J and Li L 1975 *Phys. Rev.* D **12** 3583
Magyari E 1980 *Phys. Lett.* **95B** 295

Jena S and Tripati T 1983 *Phys. Rev.* D **28** 1780
[5] Long C and Robson D 1983 *Phys. Rev.* D **27** 644
[6] McEnnan J, Botto D J and Pratt R H 1977 *Phys. Rev.* A **16** 1768
Gabriel O V, Chaudhuri S and Pratti R H 1981 *Phys. Rev.* A **24** 3088
Vrscay E R and Hamidian H 1988 *Phys. Lett.* **130A** 141
[7] Rogers G W 1984 *Phys. Rev.* A **30** 35
Au C K and Aharonov Y 1979 *Phys. Rev.* A **20** 2245
[8] Mustafa O and Barakat T 1997 *Commun. Theor. Phys.* **28** 257
[9] Mustafa O and Barakat T 1998 *Commun. Theor. Phys.* **29** accepted for publication
[10] Mustafa O 1993 *J. Phys.: Condens. Matter* **5** 1327
[11] Imbo T Pagnamenta N and Sukhatme U 1984 *Phys. Rev.* D **29** 1669
Mustafa O and Chhajlany S C 1994 *Phys. Rev.* A **50** 2926
[12] Panja M, Dutt R and Varshni Y P 1990 *Phys. Rev.* A **42** 106
Panja M, Bag M, Dutt R and Varshni Y P 1992 *Phys. Rev.* A **45** 1523
Roychoudhury R and Varshni Y P 1989 *Phys. Rev.* A **39** 5523
[13] McQuarrie B R and Vrscay E R 1993 *Phys. Rev.* A **47** 868
[14] Swenson R J and Danforth S H 1972 *J. Chem. Phys.* **57** 1734
Killingbeck J 1978 *Phys. Lett.* **65A** 87
Fock V 1971 *Z. Phys.* **61** 126
McKinley W A 1971 *Am. J. Phys.* **39** 905
[15] Epstein S 1976 *Am. J. Phys.* **44** 251
McRae S M and Vrscay E R 1992 *J. Math. Phys.* **33** 3004
Lai C S 1982 *J. Physique* A **15** L155
[16] Bednar M 1973 *Ann. Phys., NY* **75** 305
Cizek J and Vrscay E R 1982 *Int. J. Quantum Chem.* **21** 27
[17] Schiff L I 1955 *Quantum Mechanics* 2nd edn (New York: McGraw-Hill)
[18] Ikhdair S, Mustafa O and Sever R 1993 *Hadronic J.* **16** 57
[19] Greiner W 1990 *Relativistic Quantum Mechanics* (Berlin: Springer)
[20] Baeyer H C 1975 *Phys. Rev.* D **12** 3086